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An $\hbar$-deformation of the W_N algebra and its vertex operators

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Abstract. In this paper, we derive an $\hbar$-deformation of the W_N algebra and its quantum Miura transformation. The vertex operators for this $\hbar$-deformed W_N algebra and its commutation relations are also obtained.

1. Introduction

Recently, the studies of q-deformation of some infinite-dimensional algebra—q-deformed affine algebra [4, 6, 11], q-deformed Virasoro [2, 21] algebra and W_N-algebra [2, 3, 9, 10]—have attracted much attention from physicist and mathematicians. The q-deformed affine algebra and its vertex operators provide a powerful method to study the state space and the correlation function of the solvable lattice model both in the bulk case [16] and the boundary case [14]. However, the symmetry of q-deformed affine algebra only corresponds to the current algebra (affine Lie algebra) symmetry, not to Virasoro-and W-algebra-type symmetry, in conformal field theory (CFT). The q-deformation of Virasoro and W algebra, which it is thought would play the role of symmetry algebra for the solvable lattice model, has been expected for a long time. Awata *et al* [2] also constructed the q-deformed W_N algebra (including Virasoro algebra) and the associated Miura transformation from a study of the Macdonald symmetrical functions. On the other hand, Frenkel and Reshetikhin [10] succeeded in constructing the q-deformed classical W_N algebra and the corresponding Miura transformation in an analysis of the $U_q(\hat{sl}_N)$ algebra at the critical level. Feigin and Frenkel [9] then obtained the quantum version of this q-deformed classical W_N algebra, i.e. the q-deformed W_N algebra. The q-deformed Virasoro algebra has also been given by Lukyanov and Pugai [21] in studying the bosonization for the ABF (Andrews–Baxter–Forrester) model. The bosonization for vertex operators of q-deformed Virasoro [17, 21] and W_N algebra [1, 2] have been constructed.

However, there exists another important deformation of infinite-dimensional algebra, which plays an important role in completely integrable field theories (in order to make a comparison with q-deformation, we call it $\hbar$-deformation). This deformation for affine algebra was created by Drinfeld [7] in studies of the Yangian. It has been shown that the Yangian ($DY(\hat{sl}_2)$) is the dynamical non-Abelian symmetry algebra for the SU(2)-invariant Thirring model [13, 18, 19, 23]. Naively, the $\hbar$-deformed affine algebra (or Yangian) would be expected to play the same role in integrable field theories as the q-deformed affine algebra

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in the solvable lattice model. Naturally, the $\bar{h}$-deformed Virasoro and W algebra, which would play the role of symmetry algebra of some integrable field model, are expected. We have succeeded in constructing the $\bar{h}$-deformed Virasoro algebra in [12] and shown that this $\bar{h}$-deformed Virasoro algebra is the dynamical symmetry algebra of the restricted sine–Gordon model. In this paper, we construct the $\bar{h}$-deformed W_N algebra (including the $\bar{h}$-deformed Virasoro algebra as a special case of $N = 2$), the corresponding quantum Miura transformation and its vertex operators. The $\bar{h}$-deformed W_N algebra becomes the usual non-deformed W_N algebra [8] with some centre charge which is related to parameter ξ, when $\bar{h} \longrightarrow 0$ and ξ and β are fixed.

This paper is arranged as follows. In section 2, we define the $\bar{h}$-deformed W_N algebra and its Miura transformation. The screening currents and vertex operators are derived in sections 3 and 4.

2. $\bar{h}$-deformation of W_N algebra

In this section, we start by defining an $\bar{h}$-deformed W_N algebra via the quantum Miura transformation.

2.1. $A^{(1)}_{N-1}$-type weight

In this subsection, we shall give some notation about the $A^{(1)}_{N-1}$-type weight which will be used in the following parts of this paper. Let $\epsilon_\mu (1 \leqslant \mu \leqslant N)$ be the orthonormal basis in $\mathbb{R}^N$, which is supplied with the inner product $\langle \epsilon_\mu, \epsilon_\nu \rangle = \delta_{\mu\nu}$. Set

$$\bar{\epsilon}_\mu = \epsilon_\mu - \epsilon \qquad \epsilon = \frac{1}{N} \sum_{\mu=1}^{N} \epsilon_\mu. \tag{1}$$

The $A^{(1)}_{N-1}$ type weight lattice is the linear space of

$$P = \sum_{\mu=1}^{N} Z \bar{\epsilon}_\mu.$$

Note that $\sum_{\mu=1}^{N} \bar{\epsilon}_\mu = 0$. Let $\omega_\mu (1 \leqslant \mu \leqslant N-1)$ be the fundamental weights

$$\omega_\mu = \sum_{\nu=1}^{\mu} \bar{\epsilon}_\nu$$

and α_μ the simple roots $(1 \leqslant \mu \leqslant N-1)$

$$\alpha_\mu = \bar{\epsilon}_\mu - \bar{\epsilon}_{\mu+1} = \epsilon_\mu - \epsilon_{\mu+1}. \tag{2}$$

An ordered pair $(b, a) \in \mathbb{P}^2$ is called admissible if only if there exists $\mu \in (1 \leqslant \mu \leqslant N-1)$ such that

$$b - a = \bar{\epsilon}_\mu.$$

An ordered set of four weights $\begin{pmatrix} c & d \\ b & a \end{pmatrix} \in \mathbb{P}^4$ is called an admissible configuration around a face if and only if the pairs (b, a), (c, b), (d, a) and (c, d) are all admissible pairs. To each admissible configuration around a face we shall associate a Boltzmann weight in section 4.

2.2. Quantum Miura transformation

Let us consider free bosons $\lambda_i(t)$ $(i = 1, \ldots, N)$ with a continuous parameter $t \in \{\mathbb{R} - 0\}$ which satisfy

$$[\lambda_i(t), \lambda_i(t')] = \frac{4\mathrm{sh}((N-1)\overline{h}t/2)\mathrm{sh}(\overline{h}\xi t/2)\mathrm{sh}(\overline{h}(\xi+1)t/2)}{t\,\mathrm{sh}(N\overline{h}t/2)}\delta(t+t') \tag{3}$$

$$[\lambda_i(t), \lambda_j(t')] = -\frac{4\mathrm{sh}(\overline{h}t/2)\mathrm{sh}(\overline{h}\xi t/2)\mathrm{sh}(\overline{h}(\xi+1)t/2)\,\mathrm{e}^{\mathrm{sign}(j-i)N\overline{h}t/2}}{t\,\mathrm{sh}(N\overline{h}t/2)}\delta(t+t') \qquad i \neq j \tag{4}$$

with the deformed parameter $\overline{h}$ and a generic parameter ξ, where $\lambda_i(t)$ is subject to the following condition:

$$\sum_{l=1}^{N} \lambda_l(t)\,\mathrm{e}^{l\overline{h}t} = 0. \tag{5}$$

One can check that the restricted condition is compatable with equations (3) and (4).

Remark. The free bosons with continuous parameter in the case of $N = 2$, were first introduced by Jimbo *et al* [15] when studying the massless XXZ mode. This kind of bosons could be used to construct the bosonization of a Yangian double with centre $DY(\hat{sl}_N)$.

Let us define the fundamental operators $\Lambda_i(\beta)$ and the $\overline{h}$-deformed W_N algebra generators $T_i(\beta)$ for $i = 1, \ldots, N$ as follows,

$$\Lambda_i(\beta) =: \exp\left\{-\int_{-\infty}^{\infty} \lambda_i(t)\,\mathrm{e}^{\mathrm{i}\beta t}\,\mathrm{d}t\right\}: \tag{6}$$

$$T_l(\beta) = \sum_{1\leqslant j_1<j_2<\cdots<j_l\leqslant N} :\Lambda_{j_1}\left(\beta + \mathrm{i}\frac{l-1}{2}\overline{h}\right)\Lambda_{j_2}\left(\beta + \mathrm{i}\frac{l-3}{2}\overline{h}\right)\ldots\Lambda_{j_l}\left(\beta - \mathrm{i}\frac{l-1}{2}\overline{h}\right): \tag{7}$$

and $T_0(\beta) = 1$. Here $:O:$ stands for the usual bosonic normal ordering of some operator O such that the bosons $\lambda_i(t)$ with non-negative mode $t > 0$ are in the right. The restricted condition for bosons $\lambda_i(t)$ in equation (5) results in $T_N(\beta) = 1$. Actually, the generators $T_i(\beta)$ are obtained by the following quantum Miura transformation:

$$\begin{aligned}&:(\mathrm{e}^{\mathrm{i}\overline{h}\partial_\beta} - \Lambda_1(\beta))(\mathrm{e}^{\mathrm{i}\overline{h}\partial_\beta} - \Lambda_2(\beta - \mathrm{i}\overline{h}))\ldots(\mathrm{e}^{\mathrm{i}\overline{h}\partial_\beta} - \Lambda_N(\beta - \mathrm{i}(N-1)\overline{h})):\\ &\qquad = \sum_{l=0}^{N}(-1)^l T_l\left(\beta - \mathrm{i}\frac{l-1}{2}\overline{h}\right)\mathrm{e}^{\mathrm{i}(N-l)\overline{h}\partial_\beta}.\end{aligned} \tag{8}$$

Remark. $\mathrm{e}^{\mathrm{i}\overline{h}\partial_\beta}$ is the $\overline{h}$-shift operator such that

$$\mathrm{e}^{\mathrm{i}\overline{h}\partial_\beta} f(\beta) = f(\beta + \mathrm{i}\overline{h}).$$

If we take the limit of $\xi \longrightarrow -1$, the above generators $T_l(\beta)$ reduce to the classical version of the $\overline{h}$-deformed W_N algebra, which can be obtained by studying the Yangian double with centre $DY(\hat{sl}_N)$ at the critical level (i.e. $l = -N$). For the case of $N = 2$, the corresponding classical $\overline{h}$-deformed W_2 (Virasoro) algebra has been given by Ding *et al* [5]. Moreover, for the general case of $2 \leqslant N$, the corresponding classical $\overline{h}$-deformed W_N algebra has been obtained by Hou and Yang [24]. Thus, we call the limit ($\xi \longrightarrow -1$ with $\overline{h}$ and β fixed) the classical limit.

Let us consider another limit: $\overline{h} \longrightarrow 0$ with fixed ξ. Then we have $\Lambda_i(\beta) = 1 + \mathrm{i}\overline{h}\chi_i(\beta) + o(\overline{h})$ and $\mathrm{e}^{\mathrm{i}\overline{h}\partial_\beta} = 1 + \mathrm{i}\overline{h}\partial_\beta + o(\overline{h})$. Hence the right-hand side of (8) in this limit becomes

$$:(\mathrm{i}\overline{h})^N(\partial_\beta - \chi_1(\beta))(\partial_\beta - \chi_2(\beta))\ldots(\partial_\beta - \chi_N(\beta)): + o(\overline{h}^N) \tag{9}$$

and we obtain the normally ordered Miura transformation corresponding to the non-deformed W_N algebra introduced by Fateev and Lukyanov [8]. Therefore, the non-deformed W_N algebra (the ordinary one) with the centre charge $(N-1) - (N(N+1)/\xi(1+\xi))$ can be obtained by taking this kind of limit. In this sense, we call this limit ($\overline{h} \longrightarrow 0$ with fixed ξ and β) the conformal limit.

2.3. Relations of the $\overline{h}$-deformed W_N algebra

In order to obtain the commutation relations for bosonic operators, we should make a comment about regularization. When one computes the exchange relation of bosonic operators, one often encounters an integral

$$\int_0^\infty F(t)\,\mathrm{d}t$$

which is divergent at $t = 0$. Hence we adopt the regularization given by Jimbo *et al* [15]. Namely, the above integral should be understood as the contour integral

$$\int_C F(t)\frac{\log(-t)}{2\mathrm{i}\pi}\,\mathrm{d}t \tag{10}$$

where the contour C is chosen as the same as that in [15]. From the definition of fundamental operators $\Lambda_i(\beta)$ and the commutation relations of bosons $\lambda_i(t)$, we can derive the following OPEs (operator product equations):

$$\Lambda_i(\beta_1)\Lambda_i(\beta_2) = \phi_{i=i}(\beta_2 - \beta_1) : \Lambda_i(\beta_1)\Lambda_i(\beta_2) : \tag{11}$$

$$\Lambda_i(\beta_1)\Lambda_j(\beta_2) = \phi_{i<j}(\beta_2 - \beta_1) : \Lambda_i(\beta_1)\Lambda_i(\beta_2) : \qquad i < j \tag{12}$$

$$\Lambda_i(\beta_1)\Lambda_j(\beta_2) = \phi_{i>j}(\beta_2 - \beta_1) : \Lambda_i(\beta_1)\Lambda_j(\beta_2) : \qquad i > j \tag{13}$$

$$\begin{aligned}
\phi_{i=i}(\beta) &= \left[\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} - \frac{\xi}{N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + 1 - \frac{1}{N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + \frac{1+\xi}{N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + 1\right)\right] \\
&\quad \times \left[\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} - \frac{1+\xi}{N} + 1\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + \frac{1}{N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + 1 + \frac{\xi}{N}\right)\right]^{-1} \\
\phi_{i<j}(\beta) &= \left[\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} - \frac{1}{N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} - \frac{\xi}{N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + \frac{1+\xi}{N}\right)\right] \\
&\quad \times \left[\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} - \frac{1+\xi}{N}\right)\Gamma\left(\frac{i\beta}{N\overline{h}} + \frac{\xi}{N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + \frac{1}{N}\right)\right]^{-1} \\
\phi_{i>j}(\beta) &= \left[\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + 1 - \frac{1}{N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + 1 - \frac{\xi}{N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + 1 + \frac{1+\xi}{N}\right)\right] \\
&\quad \times \left[\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + 1 - \frac{1+\xi}{N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + 1 + \frac{\xi}{N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}} + 1 + \frac{1}{N}\right)\right]^{-1}.
\end{aligned} \tag{14}$$

To calculate the general OPEs, the integral representation for the Γ-function is very useful:

$$\Gamma(z) = \exp\left\{\int_0^\infty \left(\frac{\mathrm{e}^{-zt} - \mathrm{e}^{-t}}{1 - \mathrm{e}^{-t}} + (z-1)\,\mathrm{e}^{-t}\right)\frac{\mathrm{d}t}{t}\right\} \qquad \mathrm{Re}(z) > 0. \tag{15}$$

Remark. The above OPEs can be considered as the operator scaling limit of q-deformed $\Lambda_i(z)$ given by Awata *et al* [2] or by Feigin and Frenkel [9] in studying the q-deformed W_N algebra. The scaling limit is taken as as follows:

$$z = p^{-\mathrm{i}\beta/\overline{h}} \qquad q = p^{-\xi} \qquad \Lambda_i(\beta) = \lim_{p\longrightarrow 1} \Lambda_i(z) \equiv \lim_{p\longrightarrow 1} \Lambda_i(p^{(-\mathrm{i}\beta/\overline{h})}). \tag{16}$$

Theorem 1. The generators $T_1(\beta)$ and $T_m(\beta)$ of the $\overline{h}$-deformed W_N algebra satisfy the following relations,

$$\begin{aligned} & f_{1m}^{-1}(\beta_2-\beta_1)T_1(\beta_1)T_m(\beta_2) - f_{1m}^{-1}(\beta_1-\beta_2)T_m(\beta_2)T_1(\beta_1) \\ &\quad = -2\mathrm{i}\pi\left\{\mathrm{i}\overline{h}\xi(\xi+1)\left(T_{m+1}\left(\beta_2+\frac{\mathrm{i}\overline{h}}{2}\right)\delta\left(\beta_1-\beta_2-\mathrm{i}\frac{m+1}{2}\overline{h}\right)\right.\right. \\ &\qquad \left.\left. -T_{m+1}\left(\beta_2-\frac{\mathrm{i}\overline{h}}{2}\right)\delta\left(\beta_1-\beta_2+\mathrm{i}\frac{m+1}{2}\overline{h}\right)\right)\right\} \end{aligned} \tag{17}$$

where

$$\begin{aligned} f_{1m}(\beta) = & \left[\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}}+1-\frac{1+m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}}+1+\frac{1-m}{2N}\right)\right. \\ & \left.\times\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}}-\frac{\xi}{N}-\frac{1-m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}}+\frac{\xi}{N}+\frac{1+m}{2N}\right)\right] \\ & \times\left[\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}}+1-\frac{\xi}{N}-\frac{1+m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}}+1+\frac{\xi}{N}+\frac{1-m}{2N}\right)\right. \\ & \left.\times\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}}-\frac{1-m}{2N}\right)\Gamma\left(\frac{\mathrm{i}\beta}{N\overline{h}}+\frac{1+m}{2N}\right)\right]^{-1}. \end{aligned} \tag{18}$$

Proof. Using the OPEs. (11)–(13), we obtain that when $\mathrm{Im}\beta_2 \ll \mathrm{Im}\beta_1$

$$\Lambda_l(\beta_1) : \Lambda_{j_1}\left(\beta_2+\mathrm{i}\frac{m-1}{2}\overline{h}\right)\ldots\Lambda_{j_m}\left(\beta_2-\mathrm{i}\frac{m-1}{2}\overline{h}\right):$$

is equal to

$$f_{1m}(\beta_2-\beta_1) : \Lambda_l(\beta_1)\Lambda_{j_1}\left(\beta_2+\mathrm{i}\frac{m-1}{2}\overline{h}\right)\ldots\Lambda_{j_m}\left(\beta_2-\mathrm{i}\frac{m-1}{2}\overline{h}\right):$$

if $l = j_k$ for some $k \in \{1,\ldots,m\}$; and

$$\begin{aligned} & f_{1m}(\beta_2-\beta_1)\left[\left(\mathrm{i}\frac{\beta_2-\beta_1}{N\overline{h}}-\frac{\xi}{N}-\frac{1}{2N}+\frac{2k-m}{2N}\right)\left(\mathrm{i}\frac{\beta_2-\beta_1}{N\overline{h}}+\frac{\xi}{N}+\frac{1}{2N}+\frac{2k-m}{2N}\right)\right] \\ &\quad \times\left[\left(\mathrm{i}\frac{\beta_2-\beta_1}{N\overline{h}}-\frac{1}{2N}+\frac{2k-m}{2N}\right)\left(\mathrm{i}\frac{\beta_2-\beta_1}{N\overline{h}}+\frac{1}{2N}+\frac{2k-m}{2N}\right)\right]^{-1} \\ &\quad \times : \Lambda_l(\beta_1)\Lambda_{j_1}\left(\beta_2+\mathrm{i}\frac{m-1}{2}\overline{h}\right)\ldots\Lambda_{j_m}\left(\beta_2-\mathrm{i}\frac{m-1}{2}\overline{h}\right): \end{aligned}$$

if $j_k < l < j_{k+1}$. Here and in the following case $l < j_1$ corresponds to $k = 0$ and the case $l > j_m$ corresponds to $k = m$. On the other hand, when $\mathrm{Im}\beta_2 \gg \mathrm{Im}\beta_1$,

$$: \Lambda_{j_1}\left(\beta_2+\mathrm{i}\frac{m-1}{2}\overline{h}\right)\ldots\Lambda_{j_m}\left(\beta_2-\mathrm{i}\frac{m-1}{2}\overline{h}\right) : \Lambda_l(\beta_1)$$

is equal to

$$f_{1m}(\beta_1-\beta_2) : \Lambda_{j_1}\left(\beta_2+\mathrm{i}\frac{m-1}{2}\overline{h}\right)\ldots\Lambda_{j_m}\left(\beta_2-\mathrm{i}\frac{m-1}{2}\overline{h}\right)\Lambda_l(\beta_1):$$

if $l = j_k$ for some $k \in \{1, \ldots, m\}$; and

$$f_{1m}(\beta_1-\beta_2)\left[\left(\mathrm{i}\frac{\beta_1-\beta_2}{N\hbar}-\frac{\xi}{N}-\frac{1}{2N}-\frac{2k-m}{2N}\right)\left(\mathrm{i}\frac{\beta_1-\beta_2}{N\hbar}+\frac{\xi}{N}+\frac{1}{2N}-\frac{2k-m}{2N}\right)\right]$$
$$\times\left[\left(\mathrm{i}\frac{\beta_1-\beta_2}{N\hbar}-\frac{1}{2N}-\frac{2k-m}{2N}\right)\left(\mathrm{i}\frac{\beta_1-\beta_2}{N\hbar}+\frac{1}{2N}-\frac{2k-m}{2N}\right)\right]^{-1}$$
$$\times :\Lambda_{j_1}\left(\beta_2+\mathrm{i}\frac{m-1}{2}\hbar\right)\ldots\Lambda_{j_m}\left(\beta_2-\mathrm{i}\frac{m-1}{2}\hbar\right)\Lambda_l(\beta_1):$$

if $j_k < l < j_{k+1}$. Noting that

$$\lim_{\epsilon\longrightarrow 0+}\left(\frac{1}{x+\mathrm{i}\epsilon}-\frac{1}{x-\mathrm{i}\epsilon}\right)=-2\mathrm{i}\pi\delta(x) \tag{19}$$

we can obtain the commutation relations (17) for $T_1(\beta_1)$ and $T_m(\beta_2)$ after some straightforward calculations. Therefore, we complete the proof of theorem 1. □

In fact, the commutation relations for the generators of the $\hbar$-deformed W_N algebra have already been defined from the commutation relations of the fundamental operators $\Lambda_i(\beta)$ in equation (6) and the corresponding quantum Miura transformation in (7). So, one can also derive some similar commutation relations between $T_i(\beta)$ and $T_j(\beta)$ with $i, j > 1$ using the same method as that in the proof of theorem 1. These commutation relations are quadratic, and involve products of $T_{i-r}(\beta)$ and $T_{j+r}(\beta)$ with $r = 1, \ldots, \min(i, j) - 1$.

In the case of $N = 2$, this $\hbar$-deformed W_2 algebra becomes an $\hbar$-deformed Virasoro algebra, which we have studied in [12]. Here, we give an example for the case $N = 3$. The generators of this case are

$$T_1(\beta)=\Lambda_1(\beta)+\Lambda_2(\beta)+\Lambda_3(\beta) \tag{20}$$

$$T_2(\beta)=:\Lambda_1\left(\beta+\frac{\mathrm{i}\hbar}{2}\right)\Lambda_2\left(\beta-\frac{\mathrm{i}\hbar}{2}\right):+:\Lambda_1\left(\beta+\frac{\mathrm{i}\hbar}{2}\right)\Lambda_3\left(\beta-\frac{\mathrm{i}\hbar}{2}\right):$$
$$+:\Lambda_2\left(\beta+\frac{\mathrm{i}\hbar}{2}\right)\Lambda_3\left(\beta-\frac{\mathrm{i}\hbar}{2}\right):. \tag{21}$$

The commutation relations for these two generators are

$$f_{11}^{-1}(\beta_2-\beta_1)T_1(\beta_1)T_1(\beta_2)-f_{11}^{-1}(\beta_1-\beta_2)T_1(\beta_2)T_1(\beta_1)$$
$$=-2\mathrm{i}\pi\left\{\mathrm{i}\hbar\xi(\xi+1)\left(T_2\left(\beta_2+\frac{\mathrm{i}\hbar}{2}\right)\delta(\beta_1-\beta_2-\mathrm{i}\hbar)\right.\right.$$
$$\left.\left.-T_2\left(\beta_2-\frac{\mathrm{i}\hbar}{2}\right)\delta(\beta_1-\beta_2+\mathrm{i}\hbar)\right)\right\} \tag{22}$$

$$f_{12}^{-1}(\beta_2-\beta_1)T_1(\beta_1)T_2(\beta_2)-f_{12}^{-1}(\beta_1-\beta_2)T_2(\beta_2)T_1(\beta_1)$$
$$=-2\mathrm{i}\pi\left\{\mathrm{i}\hbar\xi(\xi+1)\left(\delta\left(\beta_1-\beta_2-\mathrm{i}\frac{3}{2}\hbar\right)-\delta\left(\beta_1-\beta_2+\mathrm{i}\frac{3}{2}\hbar\right)\right)\right\} \tag{23}$$

$$f_{22}^{-1}(\beta_2-\beta_1)T_2(\beta_1)T_2(\beta_2)-f_{22}^{-1}(\beta_1-\beta_2)T_2(\beta_2)T_2(\beta_1)$$
$$=-2\mathrm{i}\pi\left\{\mathrm{i}\hbar\xi(\xi+1)\left(T_1\left(\beta_2+\frac{\mathrm{i}\hbar}{2}\right)\delta(\beta_1-\beta_2-\mathrm{i}\hbar)\right.\right.$$
$$\left.\left.-T_1\left(\beta_2-\frac{\mathrm{i}\hbar}{2}\right)\delta(\beta_1-\beta_2+\mathrm{i}\hbar)\right)\right\} \tag{24}$$

where the coefficient function $f_{ij}(\beta)$ are

$$f_{11}(\beta) = \left[\Gamma\left(\frac{i\beta}{N\overline{h}}+1-\frac{1}{N}\right)\Gamma\left(\frac{i\beta}{N\overline{h}}+1\right)\Gamma\left(\frac{i\beta}{N\overline{h}}-\frac{\xi}{N}\right)\Gamma\left(\frac{i\beta}{N\overline{h}}+\frac{\xi}{N}+\frac{1}{N}\right)\right]$$
$$\times\left[\Gamma\left(\frac{i\beta}{N\overline{h}}+1-\frac{\xi}{N}-\frac{1}{N}\right)\Gamma\left(\frac{i\beta}{N\overline{h}}+1+\frac{\xi}{N}\right)\Gamma\left(\frac{i\beta}{N\overline{h}}\right)\Gamma\left(\frac{i\beta}{N\overline{h}}+\frac{1}{N}\right)\right]^{-1}$$
$$f_{12}(\beta) = \left[\Gamma\left(\frac{i\beta}{N\overline{h}}+1-\frac{3}{2N}\right)\Gamma\left(\frac{i\beta}{N\overline{h}}+1-\frac{1}{2N}\right)\right.$$
$$\left.\times\Gamma\left(\frac{i\beta}{N\overline{h}}-\frac{\xi}{N}+\frac{1}{2N}\right)\Gamma\left(\frac{i\beta}{N\overline{h}}+\frac{\xi}{N}+\frac{3}{2N}\right)\right]$$
$$\times\left[\Gamma\left(\frac{i\beta}{N\overline{h}}+1-\frac{\xi}{N}-\frac{3}{2N}\right)\Gamma\left(\frac{i\beta}{N\overline{h}}+1+\frac{\xi}{N}-\frac{1}{2N}\right)\right.$$
$$\left.\times\Gamma\left(\frac{i\beta}{N\overline{h}}+\frac{1}{2N}\right)\Gamma\left(\frac{i\beta}{N\overline{h}}+\frac{3}{2N}\right)\right]^{-1}$$
$$f_{11}(\beta) = f_{22}(\beta).$$

3. Screening currents

In this section, we will consider the screening currents for the $\overline{h}$-deformed W_N algebra. First, we introduce some zero mode operators. To each vector $\alpha \in \mathbb{P}$ (the $A_{N-1}^{(1)}$-type weight lattice defined in section 2.1), we associate operators P_α and Q_α which satisfy

$$[iP_\alpha, Q_\beta] = \langle\alpha, \beta\rangle \qquad (\alpha, \beta \in \mathbb{P}). \tag{25}$$

We shall deal with the bosonic Fock spaces $F_{l,k}(l, k \in \mathbb{P})$ generated by $\lambda_i(-t)(t > 0)$ over the vacuum states $|l, k\rangle$ .The vacuum states $|l, k\rangle$ are defined by

$$\lambda_i(t)|l, k\rangle = 0 \qquad \text{if } t > 0$$
$$P_\beta|l, k\rangle = \langle\beta, \alpha_+ l + \alpha_- k\rangle|l, k\rangle$$
$$|l, k\rangle = e^{i\alpha_+ Q_l + i\alpha_- Q_k}|0, 0\rangle$$

where $\alpha_\pm$ are some parameters related to ξ

$$\alpha_+ = -\sqrt{\frac{1+\xi}{\xi}} \qquad \alpha_- = \sqrt{\frac{\xi}{1+\xi}} \tag{26}$$

and we also introduce α_0,

$$\alpha_0 = \frac{1}{\sqrt{\xi(1+\xi)}}. \tag{27}$$

To each simple root α_j $(j = 1, \ldots, N-1)$, let us introduce two series bosons $s_j^\pm(t)$ which are defined by

$$s_j^+(t) = \frac{e^{(j\overline{h}t/2)}}{2\mathrm{sh}(\xi\overline{h}t/2)}(\lambda_j(t) - \lambda_{j+1}(t)) \tag{28}$$
$$s_j^-(t) = \frac{e^{(j\overline{h}t/2)}}{2\mathrm{sh}((1+\xi)\overline{h}t/2)}(\lambda_j(t) - \lambda_{j+1}(t)). \tag{29}$$

By these simple root bosons, we can define the screening currents as follows:

$$S_j^+(\beta) =: \exp\left\{ -\int_{-\infty}^{\infty} s_j^+(t)\, e^{i\beta t}\, dt \right\} :\ e^{-i\alpha_+ Q_{\alpha_j}} \tag{30}$$

$$S_j^-(\beta) =: \exp\left\{ \int_{-\infty}^{\infty} s_j^-(t)\, e^{i\beta t}\, dt \right\} :\ e^{-i\alpha_- Q_{\alpha_j}}. \tag{31}$$

Then we have:

Theorem 2. The screening currents $S_j^+(\beta)$ satisfy

$$\begin{aligned}
&[: (e^{i\bar{h}\partial_\beta} - \Lambda_1(\beta))(e^{i\bar{h}\partial_\beta} - \Lambda_2(\beta - i\bar{h}))\ldots(e^{i\bar{h}\partial_\beta} - \Lambda_N(\beta - i(N-1)\bar{h})) :, S_j^+(\sigma)] \\
&\quad = i\bar{h}(1+\xi)\{: (e^{i\bar{h}\partial_\beta} - \Lambda_1(\beta))\ldots(e^{i\bar{h}\partial_\beta} - \Lambda_{j-1}(\beta - i(j-2)\bar{h}))D_{\sigma,i\bar{h}\xi} \\
&\quad\quad \times \left(2\pi i\delta\left(\sigma - \beta - i\frac{j+\xi}{2}\bar{h}\right) A_j^+(\sigma)\right) \\
&\quad\quad \times e^{i\bar{h}\partial_\beta}(e^{i\bar{h}\partial_\beta} - \Lambda_{j+2}(\beta - i(j+1)\bar{h}))\ldots(e^{i\bar{h}\partial_\beta} - \Lambda_N(\beta - i(N-1)\bar{h})) :
\end{aligned} \tag{32}$$

with

$$A_j^+(\sigma) =: \Lambda_j\left(\sigma - i\frac{j+\xi}{2}\bar{h}\right) S_j^+(\sigma) :$$

and operator $D_{\sigma,i\bar{h}\xi}$ being a difference operator with variable σ:

$$D_{\sigma,\eta} f(\sigma) \equiv f(\sigma) - f(\sigma+\eta).$$

Proof. From formulae (28), we obtain the following commutation relations:

$$[\lambda_j(t), s_j^+(t')] = -\frac{2\, e^{t(1-j)\bar{h}/2}\mathrm{sh}((1+\xi)\bar{h}t/2)}{t}\delta(t+t')$$

$$[\lambda_{j+1}(t), s_j^+(t')] = \frac{2\, e^{-t(1+j)\bar{h}/2}\mathrm{sh}((1+\xi)\bar{h}t/2)}{t}\delta(t+t')$$

$$[\lambda_j(t), s_l^+(t')] = 0 \qquad \text{if } |j-l| > 1.$$

From these commutation relations and formula (15), we can derive the following OPEs:

$$\Lambda_j(\beta_1)S_j^+(\beta_2) = f_{jj}^+(\beta_2 - \beta_1) : \Lambda_j(\beta_1)S_j^+(\beta_2) : \tag{33}$$

$$S_j^+(\beta_1)\Lambda_j(\beta_2) = f_{jj}^+(\beta_2 - \beta_1) : S_j^+(\beta_1)\Lambda_j(\beta_2) : \tag{34}$$

$$\Lambda_{j+1}(\beta_1)S_j^+(\beta_2) = f_{j+1j}^+(\beta_2 - \beta_1) : \Lambda_{j+1}(\beta_1)S_j^+(\beta_2) : \tag{35}$$

$$S_j^+(\beta_1)\Lambda_{j+1}(\beta_2) = f_{j+1j}^+(\beta_2 - \beta_1) : S_j^+(\beta_1)\Lambda_{j+1}(\beta_2) : \tag{36}$$

$$S_j^+(\beta_1)\Lambda_l(\beta_2) =: S_j^+(\beta_1)\Lambda_l(\beta_2) :=: \Lambda_l(\beta_2)S_j^+(\beta_1) : \qquad \text{if } |j-l| > 1 \tag{37}$$

and

$$f_{jj}^+(\beta) = \left(\frac{i\beta}{N\bar{h}} - \frac{\xi}{2N} - \frac{1}{N} + \frac{j}{2N}\right) \Big/ \left(\frac{i\beta}{N\bar{h}} - \frac{\xi}{2N} + \frac{j}{2N} + \frac{\xi}{N}\right)$$

$$f_{j+1j}^+(\beta) = \left(\frac{i\beta}{N\bar{h}} - \frac{\xi}{2N} + \frac{1+\xi}{N} + \frac{j}{2N}\right) \Big/ \left(\frac{i\beta}{N\bar{h}} - \frac{\xi}{2N} + \frac{j}{2N}\right).$$

Formula (37) implies that the left-hand side of (32) equals

$$\begin{aligned}
&: (e^{i\bar{h}\partial_\beta} - \Lambda_1(\beta))\ldots(e^{i\bar{h}\partial_\beta} - \Lambda_{j-1}(\beta - i(j-2)\bar{h})) \\
&\quad \times[(e^{i\bar{h}\partial_\beta} - \Lambda_j(\beta - i(j-1)\bar{h}))(e^{i\bar{h}\partial_\beta} - \Lambda_{j+1}(\beta - i(j)\bar{h})), S_j^+(\sigma)] \\
&\quad \times(e^{i\bar{h}\partial_\beta} - \Lambda_{j+2}(\beta - i(j+1)\bar{h}))\ldots(e^{i\bar{h}\partial_\beta} - \Lambda_N(\beta - i(N-1)\bar{h})) : .
\end{aligned} \tag{38}$$

Therefore, it is sufficient to consider the commutation relation

$$[: (e^{i\overline{h}\partial_\beta} - \Lambda_j(\beta - i(j-1)\overline{h}))(e^{i\overline{h}\partial_\beta} - \Lambda_{j+1}(\beta - i(j)\overline{h})) :, S_j^+(\sigma)].$$

According to the OPEs (33)–(36), we can derive

$$[: \Lambda_j(\beta - i(j-1)\overline{h})\Lambda_{j+1}(\beta - i(j)\overline{h}) :, S_j^+(\sigma)] = 0.$$

Now we only need to consider the commutation relations between the term $\Lambda_j(\beta - i(j-1)\overline{h}) + \Lambda_{j+1}(\beta - i(j-1)\overline{h})$ and the screening current $S_j^+(\sigma)$. From the OPEs (33)–(36), formula (19), noting that

$$: \Lambda_j\left(\beta - i\frac{j-\xi}{2}\overline{h}\right) S_j^+(\beta + i\xi\overline{h}) :=: \Lambda_{j+1}\left(\beta - i\frac{j-\xi}{2}\overline{h}\right) S_j^+(\beta) :$$

and using the same method as that in the proof of theorem 1, we have the following commutation relation:

$$\begin{aligned}&[\Lambda_j(\beta - i(j-1)\overline{h}) + \Lambda_{j+1}(\beta - i(j-1)\overline{h}), S_j^+(\sigma)]\\ &\quad = i\overline{h}(1+\xi)D_{\sigma, i\overline{h}\xi}\left(2\pi i\delta\left(\sigma - \beta - i\frac{j+\xi}{2}\overline{h}\right) : A_j^+(\sigma)\right) : .\end{aligned}$$

Therefore, equation (32) has been obtained. □

Using the same method, we have:

Theorem 3. The second series screening currents $S_j^-(\sigma)$ satisfy

$$\begin{aligned}&[: (e^{i\overline{h}\partial_\beta} - \Lambda_1(\beta))(e^{i\overline{h}\partial_\beta} - \Lambda_2(\beta - i\overline{h}))\ldots(e^{i\overline{h}\partial_\beta} - \Lambda_N(\beta - i(N-1)\overline{h})) :, S_j^-(\sigma)]\\ &\quad = i\overline{h}\xi\{: (e^{i\overline{h}\partial_\beta} - \Lambda_1(\beta))\ldots(e^{i\overline{h}\partial_\beta} - \Lambda_{j-1}(\beta - i(j-2)\overline{h}))D_{\sigma, -i\overline{h}(1+\xi)}\\ &\qquad \times\left(2\pi i\delta\left(\sigma - \beta + i\frac{-j+\xi+1}{2}\overline{h}\right) A_j^-(\sigma)\right)\\ &\qquad \times e^{i\overline{h}\partial_\beta}(e^{i\overline{h}\partial_\beta} - \Lambda_{j+2}(\beta - i(j+1)\overline{h}))\ldots(e^{i\overline{h}\partial_\beta} - \Lambda_N(\beta - i(N-1)\overline{h})) :\end{aligned} \tag{39}$$

with

$$A_j^-(\sigma) =: \Lambda_j\left(\sigma + i\frac{-j+\xi+1}{2}\overline{h}\right) S_j^-(\sigma) : .$$

Therefore, the screening currents $S_j^{\pm}(\beta)$ commute with any $\overline{h}$-deformed W_N algebra generators up to total difference.

Remark. In taking the conformal limit ($\overline{h} \longrightarrow 0$ and with ξ and β fixed), the screening currents $S_j^{\pm}(\beta)$ will become the ordinary screening current [8].

Theorem 2 and 3 imply that one can construct the intertwining operators (namely, vertex operators) for the $\overline{h}$-deformed W_N algebra using the screening currents $S_j^{\pm}(\beta)$. In the next section we shall construct the vertex operators for the $\overline{h}$-deformed W_N algebra.

4. The vertex operators and their exchange relations

In this section, we construct the type I and type II vertex operators for this $\overline{h}$-deformed W_N algebra through the two series screening currents $S_j^{\pm}(\beta)$. First, we set

$$\hat{\pi}_\mu = \alpha_0^{-1} P_{\bar{\epsilon}_\mu} \qquad \hat{\pi}_{\mu\nu} = \hat{\pi}_\mu - \hat{\pi}_\nu \tag{40}$$

and

$$\hat{\pi}_{\mu\nu} F_{l,k} = \langle \epsilon_\mu - \epsilon_\nu, -(1+\xi)l + \xi k \rangle F_{l,k}. \tag{41}$$

Note that

$$\mathrm{e}^{-\mathrm{i}\alpha_\pm Q_\gamma} \hat{\pi}_\sigma \, \mathrm{e}^{\mathrm{i}\alpha_\pm Q_\gamma} = \hat{\pi}_\sigma + \alpha_0^{-1} \alpha_\pm \langle \sigma, \gamma \rangle \tag{42}$$

and this formula is very useful for calculating commutation relations of vertex operators. To each fundamental weight of ω_j $(j = 1, \ldots, N-1)$, let us introduce two series bosons $a_j(t)$ and $a'_j(t)$ which are defined by

$$a_j = \sum_{k=1}^{j} \frac{\mathrm{e}^{-\overline{h}(j-2k+1)t/2}}{2\mathrm{sh}(\overline{h}\xi t/2)} \lambda_k(t) \qquad a'_j = \sum_{k=1}^{j} \frac{\mathrm{e}^{-\overline{h}(j-2k+1)t/2}}{2\mathrm{sh}(\overline{h}(1+\xi)t/2} \lambda_k(t) \tag{43}$$

and also define

$$U_{\omega_j}(\beta) =: \exp\left\{ \int_{-\infty}^{\infty} a_j(t)\, \mathrm{e}^{\mathrm{i}\beta t}\, \mathrm{d}t \right\} : \; \mathrm{e}^{\mathrm{i}\alpha_+ Q_{\omega_j}}$$
$$U'_{\omega_j}(\beta) =: \exp\left\{ \int_{-\infty}^{\infty} -a'_j(t)\, \mathrm{e}^{\mathrm{i}\beta t}\, \mathrm{d}t \right\} : \; \mathrm{e}^{\mathrm{i}\alpha_- Q_{\omega_j}}. \tag{44}$$

Because the vertex operators associated with each fundamental weight $\omega_j (= 2, \ldots, N-1)$ can be constructed from the skew-symmetric fusion of the basic ones $U_{\omega_1}(\beta)$ and $U'_{\omega_1}(\beta)$ [1], it is sufficient to only deal with the vertex operators corresponding to the fundamental weight ω_1. In order to calculate the exchange relations of the vertex operators, we first derive the following commutation relations:

$$[a_j(t), s_j^+(t')] = -\frac{\mathrm{sh}(\overline{h}(1+\xi)t/2)}{t\mathrm{sh}(\overline{h}\xi t/2)} \delta_{j,l} \delta(t+t') \qquad [a_j(t), s_j^-(t')] = -\frac{1}{t} \delta_{j,l} \delta(t+t')$$
$$[a'_j(t), s_j^-(t')] = -\frac{\mathrm{sh}(\overline{h}\xi t/2)}{t\mathrm{sh}(\overline{h}(1+\xi)t/2)} \delta_{j,l} \delta(t+t') \qquad [a'_j(t), s_j^+(t')] = -\frac{1}{t} \delta_{j,l} \delta(t+t')$$
$$[a_1(t), a_1(t')] = -\frac{\mathrm{sh}((N-1)\overline{h}t/2)\mathrm{sh}((1+\xi)\overline{h}t/2)}{t\mathrm{sh}(N\overline{h}t/2)\mathrm{sh}(\xi\overline{h}t/2)} \delta(t+t')$$
$$[a_1(t), a'_1(t')] = -\frac{\mathrm{sh}((N-1)\overline{h}t/2)}{t\mathrm{sh}(N\overline{h}t/2)} \delta(t+t')$$
$$[a'_1(t), a'_1(t')] = -\frac{\mathrm{sh}((N-1)\overline{h}t/2)\mathrm{sh}(\xi\overline{h}t/2)}{t\mathrm{sh}(N\overline{h}t/2)\mathrm{sh}((1+\xi)\overline{h}t/2)} \delta(t+t').$$

From the above relations, and taking the regularization in section 2.3, we can derive the following exchange relations,

$$U_{\omega_1}(\beta_1) U_{\omega_1}(\beta_2) = r_1(\beta_1 - \beta_2) U_{\omega_1}(\beta_2) U_{\omega_1}(\beta_1)$$
$$U'_{\omega_1}(\beta_1) U'_{\omega_1}(\beta_2) = r'_1(\beta_1 - \beta_2) U'_{\omega_1}(\beta_2) U'_{\omega_1}(\beta_1)$$
$$U_{\omega_1}(\beta_1) U'_{\omega_1}(\beta_2) = \tau_1(\beta_1 - \beta_2) U'_{\omega_1}(\beta_2) U_{\omega_1}(\beta_1)$$
$$S_j^+(\beta_1) S_{j+1}^+(\beta_2) = -f(\beta_1 - \beta_2, 0) S_{j+1}^+(\beta_2) S_j^+(\beta_1)$$
$$S_j^-(\beta_1) S_{j+1}^-(\beta_2) = -f'(\beta_1 - \beta_2, 0) S_{j+1}^-(\beta_2) S_j^-(\beta_1)$$

$$U_{\omega_1}(\beta_1)S_1^+(\beta_2) = -f(\beta_1-\beta_2,0)S_1^+(\beta_2)U_{\omega_1}(\beta_1)$$
$$U'_{\omega_1}(\beta_1)S_1^- = -f'(\beta_1-\beta_2,0)S_1^-(\beta_2)U'_{\omega_1}(\beta_1)$$
$$U_{\omega_1}(\beta_1)S_1^-(\beta_2) = -U_{\omega_1}(\beta_1)S_1^-(\beta_2)$$
$$U'_{\omega_1}(\beta_1)S_1^+(\beta_2) = -U'_{\omega_1}(\beta_1)S_1^+(\beta_2) \qquad (45)$$

where the fundamental function $f(\beta, w)$ and $f'(\beta, w)$ (which play a very important role in constructing the vertex operators) are defined by

$$f(\beta,w) = \sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}\xi} - \frac{1}{2\xi} - \frac{w}{\xi}\right)\Big/\sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}\xi} + \frac{1}{2\xi}\right)$$
$$f'(\beta,w) = \sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}(1+\xi)} + \frac{1}{2(1+\xi)} + \frac{w}{1+\xi}\right)\Big/\sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}(1+\xi)} - \frac{1}{2(1+\xi)}\right) \qquad (46)$$

and

$$r(\beta) = \exp\left\{-\int_0^\infty \frac{2\mathrm{sh}((N-1)\overline{h}t/2)\mathrm{sh}((1+\xi)\overline{h}t/2)\mathrm{sh}(\mathrm{i}\beta t)}{t\mathrm{sh}(N\overline{h}t/2)\mathrm{sh}(\xi\overline{h}t/2)}\,\mathrm{d}t\right\} \qquad (47)$$
$$r'(\beta) = \exp\left\{-\int_0^\infty \frac{2\mathrm{sh}((N-1)\overline{h}t/2)\mathrm{sh}(\xi\overline{h}t/2)\mathrm{sh}(\mathrm{i}\beta t)}{t\mathrm{sh}(N\overline{h}t/2)\mathrm{sh}((1+\xi)\overline{h}t/2)}\,\mathrm{d}t\right\} \qquad (48)$$
$$\tau(\beta) = \sin\pi\left(\frac{1}{2N} - \frac{\mathrm{i}\beta}{N\overline{h}}\right)\Big/\sin\pi\left(\frac{1}{2N} + \frac{\mathrm{i}\beta}{N\overline{h}}\right). \qquad (49)$$

Now let us define the type I vertex operators $Z'_\mu(\beta)$ and the type II vertex operators $Z_\mu(\beta)(\mu = 1, \ldots, N)$

$$Z_\mu(\beta) = \int_{C_1}\prod_{j=1}^{\mu-1}\mathrm{d}\eta_j\, U'_{\omega_1}(\beta)S_1^-(\eta_1)S_2^-(\eta_2)\ldots S_{\mu-1}^-(\eta_{\mu-1})\prod_{j=1}^{\mu-1}f'(\eta_j-\eta_{j-1},\hat{\pi}_{j\mu}) \qquad (50)$$
$$Z'_\mu(\beta) = \int_{C_2}\prod_{j=1}^{\mu-1}\mathrm{d}\eta_j\, U_{\omega_1}(\beta)S_1^+(\eta_1)S_2^+(\eta_2)\ldots S_{\mu-1}^+(\eta_{\mu-1})\prod_{j=1}^{\mu-1}f(\eta_j-\eta_{j-1},\hat{\pi}_{j\mu}). \qquad (51)$$

It is easy to see that the vertex operators $Z_\mu(\beta)$ and $Z'_\mu(\beta)$ are some bosonic operators intertwing the Fock spaces $F_{l,k}$

$$Z_\mu(\beta) : F_{l,k} \longrightarrow F_{l,k+\overline{\epsilon}_\mu} \qquad Z'_\mu(\beta) : F_{l,k} \longrightarrow F_{l+\overline{\epsilon}_\mu,k}. \qquad (52)$$

Here we set $\eta_0 = \beta$, the integration contour C_1 is chosen as the contour corresponding to the integration variable η_j enclosing the poles $\eta_{j-1} + \mathrm{i}\frac{1}{2}\overline{h} - \mathrm{i}\overline{h}\xi n (0 \leqslant n)$, and the other integration contour C_2 is chosen as the contour corresponding to the integration variable η_j enclosing the poles $\eta_{j-1} - \mathrm{i}\frac{1}{2}\overline{h} - \mathrm{i}(1+\xi)\overline{h}n(0 \leqslant n)$.

The constructure form of our type I (type II) vertex operators seems to be similar to that of the vertex operators for the $A_{N-1}^{(1)}$ face model given by Asai *et al* [1], but with different bosonic operators and 'coefficient parts' function $f(\beta, w)$ ($f'(\beta, w)$). Thus the same trick [1] can be used to calculate the commutation relations for our vertex operators. Using the method which was presented by Asai *et al* in appendix B of the [1], we can derive the commutation relations for vertex operators $Z_\mu(\beta)$ and $Z'_\mu(\beta)$,

$$Z'_\mu(\beta_1)Z'_\nu(\beta_2) = \sum_{\mu'\nu'}^{\overline{\epsilon}_\mu+\overline{\epsilon}_\nu=\overline{\epsilon}_{\mu'}+\overline{\epsilon}_{\nu'}} Z'_{\mu'}(\beta_2)Z'_{\nu'}(\beta_1)\hat{W}'\begin{pmatrix}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_{\nu'} \\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{pmatrix}|\beta_1-\beta_2\Big) \qquad (53)$$
$$Z_\mu(\beta_1)Z_\nu(\beta_2) = \sum_{\mu'\nu'}^{\overline{\epsilon}_\mu+\overline{\epsilon}_\nu=\overline{\epsilon}_{\mu'}+\overline{\epsilon}_{\nu'}} Z_{\mu'}(\beta_2)Z_{\nu'}(\beta_1)\hat{W}\begin{pmatrix}\hat{\pi}+\overline{\epsilon}_\mu+\overline{\epsilon}_\nu & \hat{\pi}+\overline{\epsilon}_{\nu'} \\ \hat{\pi}+\overline{\epsilon}_\nu & \hat{\pi}\end{pmatrix}|\beta_1-\beta_2\Big) \qquad (54)$$
$$Z_\mu(\beta_1)Z'_\nu(\beta_2) = Z'_\nu(\beta_2)Z'_\mu(\beta_1)\tau(\beta_1-\beta_2) \qquad (55)$$

and the braid matrices (connection matrices) $\hat{W}\begin{pmatrix} \hat{\pi}+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & \hat{\pi}+\bar{\epsilon}_{\nu'} \\ \hat{\pi}+\bar{\epsilon}_\nu & \hat{\pi} \end{pmatrix} |\beta \Big)$ and $\hat{W}'\begin{pmatrix} \hat{\pi}+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & \hat{\pi}+\bar{\epsilon}_{\nu'} \\ \hat{\pi}+\bar{\epsilon}_\nu & \hat{\pi} \end{pmatrix} |\beta \Big)$ are some functions taking values on operators $\hat{\pi}_{\mu\nu}$ such as

$$\hat{W}'\left(\begin{matrix} \hat{\pi}+2\bar{\epsilon}_\mu & \hat{\pi}+\bar{\epsilon}_\mu \\ \hat{\pi}+\bar{\epsilon}_\mu & \hat{\pi} \end{matrix} \Big|\beta\right) = r(\beta) \tag{56}$$

$$\hat{W}'\left(\begin{matrix} \hat{\pi}+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & \hat{\pi}+\bar{\epsilon}_\nu \\ \hat{\pi}+\bar{\epsilon}_\nu & \hat{\pi} \end{matrix} \Big|\beta\right) = -r(\beta)\left[\sin\frac{\pi}{\xi}\sin\pi\left(\frac{\mathrm{i}\beta}{\bar{h}\xi}-\frac{\hat{\pi}_{\mu\nu}}{\xi}\right)\Big/\sin\pi\left(\frac{\hat{\pi}_{\mu\nu}}{\xi}\right)\sin\pi\left(\frac{\mathrm{i}\beta}{\bar{h}\xi}+\frac{1}{\xi}\right)\right] \tag{57}$$

$$\hat{W}'\left(\begin{matrix} \hat{\pi}+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & \hat{\pi}+\bar{\epsilon}_\mu \\ \hat{\pi}+\bar{\epsilon}_\nu & \hat{\pi} \end{matrix} \Big|\beta\right) = r(\beta)\left[\sin\pi\left(\frac{\mathrm{i}\beta}{\bar{h}\xi}\right)\sin\pi\left(\frac{1}{\xi}+\frac{\hat{\pi}_{\mu\nu}}{\xi}\right)\Big/\sin\pi\left(\frac{\hat{\pi}_{\mu\nu}}{\xi}\right)\sin\pi\left(\frac{\mathrm{i}\beta}{\bar{h}\xi}+\frac{1}{\xi}\right)\right] \tag{58}$$

$$\hat{W}\left(\begin{matrix} \hat{\pi}+2\bar{\epsilon}_\mu & \hat{\pi}+\bar{\epsilon}_\mu \\ \hat{\pi}+\bar{\epsilon}_\mu & \hat{\pi} \end{matrix} \Big|\beta\right) = r'(\beta) \tag{59}$$

$$\hat{W}\left(\begin{matrix} \hat{\pi}+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & \hat{\pi}+\bar{\epsilon}_\nu \\ \hat{\pi}+\bar{\epsilon}_\nu & \hat{\pi} \end{matrix} \Big|\beta\right) = -r'(\beta)\left\{\left[\sin\frac{\pi}{1+\xi}\sin\pi\left(\frac{\mathrm{i}\beta}{\bar{h}(1+\xi)}+\frac{\hat{\pi}_{\mu\nu}}{1+\xi}\right)\right\}\right. \times\left\{\sin\pi\left(\frac{\hat{\pi}_{\mu\nu}}{1+\xi}\right)\sin\pi\left(\frac{\mathrm{i}\beta}{\bar{h}(1+\xi)}-\frac{1}{1+\xi}\right)\right]\Bigg\}^{-1} \tag{60}$$

$$\hat{W}\left(\begin{matrix} \hat{\pi}+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & \hat{\pi}+\bar{\epsilon}_\mu \\ \hat{\pi}+\bar{\epsilon}_\nu & \hat{\pi} \end{matrix} \Big|\beta\right) = r'(\beta)\left\{\left[\sin\pi\left(\frac{\mathrm{i}\beta}{\bar{h}(1+\xi)}\right)\sin\pi\left(\frac{1}{1+\xi}+\frac{\hat{\pi}_{\mu\nu}}{1+\xi}\right)\right\}\right. \times\left\{\sin\pi\left(\frac{\hat{\pi}_{\mu\nu}}{1+\xi}\right)\sin\pi\left(\frac{\mathrm{i}\beta}{\bar{h}(1+\xi)}-\frac{1}{1+\xi}\right)\right]\Bigg\}^{-1}. \tag{61}$$

Therefore, these connection matrices do not commute with the vertex operators and the exchange relations should be written in the same order as in equations (53) and (54). Noting that

$$r(-\beta)|_{\xi\longrightarrow 1+\xi} = r'(\beta)\Delta_N(\beta)$$
$$\Delta_N(\beta) = \sin\pi\left(\frac{\mathrm{i}\beta}{N\bar{h}}+\frac{1}{N}\right)\Big/\sin\pi\left(\frac{\mathrm{i}\beta}{N\bar{h}}-\frac{1}{N}\right)$$

we find that the matrices $\hat{W}\left(\begin{matrix} \hat{\pi}+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & \hat{\pi}+\bar{\epsilon}_{\nu'} \\ \hat{\pi}+\bar{\epsilon}_\nu & \hat{\pi} \end{matrix} \Big|\beta\right)$ and $\hat{W}'\left(\begin{matrix} \hat{\pi}+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & \hat{\pi}+\bar{\epsilon}_{\nu'} \\ \hat{\pi}+\bar{\epsilon}_\nu & \hat{\pi} \end{matrix} \Big|\beta\right)$ are related to each other as follows:

$$\hat{W}'\left(\begin{matrix} \hat{\pi}+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & \hat{\pi}+\bar{\epsilon}_{\nu'} \\ \hat{\pi}+\bar{\epsilon}_\nu & \hat{\pi} \end{matrix} \Big|-\beta\right)\Big|_{\xi\longrightarrow 1+\xi} = \Delta_N(\beta)\hat{W}\left(\begin{matrix} \hat{\pi}+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & \hat{\pi}+\bar{\epsilon}_{\nu'} \\ \hat{\pi}+\bar{\epsilon}_\nu & \hat{\pi} \end{matrix} \Big|\beta\right).$$

When $N=2$, the factor $\Delta_N(\beta)=-1$, as occurred when studying the $\bar{h}$-deformed Virasoro algebra [12]. If both sides of equations (53) and (54) are acted on the special Fock space $F_{l,k}$, noting that $l_{\mu\nu}$ and $k_{\mu\nu}$

$$l_{\mu\nu} = \langle\bar{\epsilon}_\mu-\bar{\epsilon}_\nu, l\rangle \qquad k_{\mu\nu} = \langle\bar{\epsilon}_\mu-\bar{\epsilon}_\nu, k\rangle$$

are all integer, we have

$$Z'_\mu(\beta_1)Z'_\nu(\beta_2)|_{F_{l,k}} = \sum_{\mu'\nu'}^{\bar{\epsilon}_\mu+\bar{\epsilon}_\nu=\bar{\epsilon}_{\mu'}+\bar{\epsilon}_{\nu'}} Z'_{\mu'}(\beta_2)Z'_{\nu'}(\beta_1)|_{F_{l,k}} W'\left(\begin{matrix} l+\bar{\epsilon}_\mu+\bar{\epsilon}_\nu & l+\bar{\epsilon}_{\nu'} \\ l+\bar{\epsilon}_\nu & l \end{matrix} \Big|\beta_1-\beta_2\right) \tag{62}$$

$$Z_\mu(\beta_1)Z_\nu(\beta_2)|_{F_{l,k}} = \sum_{\mu'\nu'}^{\bar\epsilon_\mu+\bar\epsilon_\nu=\bar\epsilon_{\mu'}+\bar\epsilon_{\nu'}} Z_{\mu'}(\beta_2)Z_{\nu'}(\beta_1)|_{F_{l,k}} W\begin{pmatrix} k+\bar\epsilon_\mu+\bar\epsilon_\nu & k+\bar\epsilon_{\nu'} \\ k+\bar\epsilon_\nu & k \end{pmatrix}|\beta_1-\beta_2\Bigg) \tag{63}$$

and

$$W'\begin{pmatrix} l+2\bar\epsilon_\mu & l+\bar\epsilon_\mu \\ l+\bar\epsilon_\mu & l \end{pmatrix}|\beta\Bigg) = r(\beta)$$

$$W'\begin{pmatrix} l+\bar\epsilon_\mu+\bar\epsilon_\nu & l+\bar\epsilon_\nu \\ l+\bar\epsilon_\nu & l \end{pmatrix}|\beta\Bigg)$$
$$= r(\beta)\left[\sin\frac{\pi}{\xi}\sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}\xi}+\frac{l_{\mu\nu}}{\xi}\right)\Big/\sin\pi\left(\frac{l_{\mu\nu}}{\xi}\right)\sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}\xi}+\frac{1}{\xi}\right)\right]$$

$$W'\begin{pmatrix} l+\bar\epsilon_\mu+\bar\epsilon_\nu & l+\bar\epsilon_\mu \\ l+\bar\epsilon_\nu & l \end{pmatrix}|\beta\Bigg)$$
$$= r(\beta)\left[\sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}\xi}\right)\sin\pi\left(-\frac{1}{\xi}+\frac{l_{\mu\nu}}{\xi}\right)\Big/\sin\pi\left(\frac{l_{\mu\nu}}{\xi}\right)\sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}\xi}+\frac{1}{\xi}\right)\right]$$

$$W\begin{pmatrix} k+2\bar\epsilon_\mu & k+\bar\epsilon_\mu \\ k+\bar\epsilon_\mu & k \end{pmatrix}|\beta\Bigg) = r'(\beta)$$

$$W\begin{pmatrix} k+\bar\epsilon_\mu+\bar\epsilon_\nu & k+\bar\epsilon_\nu \\ k+\bar\epsilon_\nu & k \end{pmatrix}|\beta\Bigg)$$
$$= r'(\beta)\left\{\left[\sin\frac{\pi}{1+\xi}\sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}(1+\xi)}-\frac{k_{\mu\nu}}{1+\xi}\right)\right\}\right.$$
$$\times\left\{\sin\pi\left(\frac{k_{\mu\nu}}{1+\xi}\right)\sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}(1+\xi)}-\frac{1}{1+\xi}\right)\right]\Bigg\}^{-1}$$

$$W\begin{pmatrix} k+\bar\epsilon_\mu+\bar\epsilon_\nu & k+\bar\epsilon_\mu \\ k+\bar\epsilon_\nu & k \end{pmatrix}|\beta\Bigg)$$
$$= r'(\beta)\left\{\left[\sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}(1+\xi)}\right)\sin\pi\left(-\frac{1}{1+\xi}+\frac{k_{\mu\nu}}{1+\xi}\right)\right\}\right.$$
$$\times\left\{\sin\pi\left(\frac{k_{\mu\nu}}{1+\xi}\right)\sin\pi\left(\frac{\mathrm{i}\beta}{\overline{h}(1+\xi)}-\frac{1}{1+\xi}\right)\right]\Bigg\}^{-1}.$$

It can be checked that the Boltzmann weights $W\begin{pmatrix} c & d \\ b & a \end{pmatrix}|\beta\Big)$ and $W\begin{pmatrix} c & d \\ b & a \end{pmatrix}|\beta\Big)$ satisfy the star-triangle equations (or Yang–Baxter equation)

$$\sum_g W\begin{pmatrix} d & e \\ c & g \end{pmatrix}|\beta_1\Bigg) W\begin{pmatrix} c & g \\ b & a \end{pmatrix}|\beta_2\Bigg) W\begin{pmatrix} e & f \\ g & a \end{pmatrix}|\beta_1-\beta_2\Bigg)$$
$$= \sum_g W\begin{pmatrix} g & f \\ b & a \end{pmatrix}|\beta_1\Bigg) W\begin{pmatrix} d & e \\ g & f \end{pmatrix}|\beta_2\Bigg) W\begin{pmatrix} d & g \\ c & b \end{pmatrix}|\beta_1-\beta_2\Bigg) \tag{64}$$

and unitary relation

$$\sum_g W\begin{pmatrix} c & g \\ b & a \end{pmatrix}|-\beta\Bigg) W\begin{pmatrix} c & d \\ g & a \end{pmatrix}|\beta\Bigg) = \delta_{bd}. \tag{65}$$

In fact, the Yang–Baxter equation (64) and the unitary relation (65) are direct results of the exchange relation of the vertex operators in equations (62) and (63) (the associativity of algebra $Z_\mu(\beta)$ and $Z'_\mu(\beta)$).

Remark. In fact, we have constructed the bosonization of the vertex operators for the trigonometric SOS (solid-on-solid) model of $A^{(1)}_{N-1}$ type.

5. Discussions

We have constructed an $\bar{h}$-deformed W_N algebra and its quantum Miura transformation. The $\bar{h}$-deformation of the W_N algebra can be obtained by two ways: one can first derive the classical version of the $\bar{h}$-deformed W_N algebra by studying the Yangian double with centre $DY(\hat{sl}_N)$ at the critical level, following Frenkel and Reshetikhin in their study of the $U_q(\hat{sl}_N)$ at the critical level [10], and then construct the (quantum) $\bar{h}$-deformed W_N algebra by quantizing the classical one; another way to construct the $\bar{h}$-deformed W_N algebra is by taking some scaling limit of the q-deformed W_N algebra such as equation (16). In fact, the same phenomena also occur in studying the Yangian double with centre $DY(\hat{sl}_N)$: the $DY(\hat{sl}_N)$ can be considered as some scaling limit of the $U_q(\hat{sl}_N)$ algebra.

We have only considered the $\bar{h}$-deformed W_N algebra for generic ξ. When ξ is some rational number ($\xi = p/q$, p and q are two coprime integers), the realization of the $\bar{h}$-deformed W_N algebra in the Fock space $F_{l,k}$ would be highly reducible and we have to throw out some states from the Fock space $F_{l,k}$ to obtain the irreducible component $H_{l,k}$. (Here we choose the same symbols as in [1]). For $N = 2$, the irreducible space $H_{l,k}$ (i.e. $L_{l,k}$ in [12]) can be obtained by some BRST cohomology [12]. Unfortunately, the constructure of the BRST complex and the calculation of cohomology for $3 \leqslant N$ is still an open problem.

We have also constructed the vertex operators of type I and type II. These vertex operators satisfy some Fadeev–Zamolodchikov algebra with face type Boltzmann weight as its constructure constant. In order to obtain the correlation functions as the traces of products of these vertex operators, we need to introduce a boost operator H,

$$H = \sum_{j=1}^{N-1} \int_0^\infty \frac{t^2 \mathrm{sh}(\xi \bar{h} t/2)}{\mathrm{sh}((1+\xi)\bar{h} t/2)} a_j(-t) s_j^+(t)\, \mathrm{d}t \tag{66}$$

which enjoys the property

$$\mathrm{e}^{2\bar{h}H} Z_\mu(t)\, \mathrm{e}^{-2\bar{h}H} = Z_\mu(t - 2\mathrm{i}\bar{h}) \qquad \mathrm{e}^{2\bar{h}H} Z'_\mu(t)\, \mathrm{e}^{-2\bar{h}H} = Z'_\mu(t - 2\mathrm{i}\bar{h}).$$

Moreover, using the skew-symmetric fusion of N vertex operators, one can obtain some invertibility for our vertex operators of the form such as (3.19) and (c.20) in [1]. Then the correlation function can be described by the following trace function:

$$G(\beta_1, \ldots, \beta_{Nn})_{\mu_1,\ldots,\mu_{Nn}} = \frac{\mathrm{tr}(\mathrm{e}^{-2\bar{h}H} Z'_{\mu_1}(\beta_1) \ldots Z'_{\mu_{Nn}}(\beta_{Nn}))}{\mathrm{tr}(\mathrm{e}^{-2\bar{h}H})}. \tag{67}$$

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